Correct Definition of the Poynting Vector in Electrically and Magnetically Polarizable Medium Reveals that Negative Refraction is Impossible

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I compute from first principles the local heating rate q (the amount of electromagnetic energy converted to heat per unit time per unit volume) for electromagnetic waves propagating in magnetically and electrically polarizable media. I find that, in magnetic media, this rate has two separate contributions, $q^{(V)}$ and $q^{(S)}$, the first coming from the volume of the medium and the second from its surface. I argue that the second law of thermodynamics requires that the volume contribution be positive and that this requirement, in turn, prohibits negative refraction. This result holds for active or passive media and in the presence of anisotropy and spatial dispersion.

1. Introduction

Macroscopic electromagnetic theory of material media which can simultaneously support electric and magnetic polarizations denoted by $\mathbf{P}$ and $\mathbf{M}$, respectively, has been developed over a century ago and is exposed in many standard textbooks. However, in the optical frequency range and at higher frequencies, this theory has long been viewed as purely abstract and nonempirical. Even at much lower frequencies, materials which simultaneously exhibit nonzero magnetic and electric susceptibilities (and are sufficiently transparent to allow any noticeable penetration of electromagnetic field into their interior) are quite rare and exotic.

While it is possible to argue about the physical attainability of artificial materials with nonzero electric and magnetic susceptibilities in any given frequency range, nothing precludes us from formally developing the electrodynamics of such media based on the macroscopic Maxwell equations. In particular, this approach was adopted by Veselago in the now famous paper (Ref. 1). Veselago was interested in materials whose electric permittivity ϵ and magnetic permeability μ are simultaneously negative and which can exhibit the so-called negative refraction - the physical effect which takes place when an electromagnetic wave entering the medium (e.g., from vacuum) is refracted at the "negative" Snell's angle.

After the publication of a more recent paper by Pendry in which a perfect (subwavelength-focusing) lens built from a negatively-refracting material was proposed [2], enormous attention was attracted to negative refraction. Numerous proposals for manufacturing artificial materials with negative refraction have been put forth. There has also been a rigorous effort to demonstrate negative refraction experimentally; see, for example, Refs. 3–5 and references therein.

Simultaneously with the activities mentioned above, there has also been a persistent effort to subject the physical attainability of negative refraction to doubt. Perhaps, the most consequential of such exploits is the recent paper by Stockman [6] in which it is shown from the causality principle that, in a negatively-refracting material, the rate of dissipation of the electromagnetic energy into heat can not be lower than a certain threshold and that any attempt to compensate for such dissipation, e.g., by introducing optical gain, will necessarily destroy the negative refraction. It is worthwhile to note that low dissipative losses are essential for the realization of the original Pendry's proposal for the perfect lens.

In this article I confront the phenomenon of negative refraction with another fundamental physical principle, the second law of thermodynamics. I show that the general requirement for negative refraction in isotropic media, namely,

$$\mathrm{Im}(\epsilon\mu) < 0 \tag{1}$$

is in contradiction with the latter. To do so, I compute the heating rate $q(\mathbf{r})$ in a magnetically and electrically polarizable medium. I do this by two methods, one involving the expression $-\nabla \cdot \mathbf{S}$, where $\mathbf{S}$ is the Poynting vector, and the other involving the expression $\mathbf{J}\cdot\mathbf{E}$, where $\mathbf{E}$ is the electric field and

$$\mathbf{J} = \frac{\partial \mathbf{P}}{\partial t} + c\nabla \times \mathbf{M} \tag{2}$$

is the total current induced in the medium (we assume that there are no *external* currents or charges). Quite unexpectedly, I obtain different results. I claim that the explanation for this discrepancy is that the Poynting vector in a magnetically polarizable medium must be defined by

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{B} \tag{3}$$

rather than by the commonly used formula

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{H}\ . \tag{4}$$

Arguments for the validity of (3) are given below.

When the definition (3) is adopted, the two methods of computing $q(\mathbf{r})$ give the same result. It further turns out that the heating rate has two separate contributions: one coming from the volume and the other from the surface of the medium. These contributions are denoted by $q^{(V)}$ and $q^{(S)}$ below. The total (that is, integral over the body volume) heat absorbed per unit time is given by the formula

$$Q = \int_V q^{(V)}(\mathbf{r})d^3r + \oint_S q^{(S)}(\mathbf{r})d^2r \ , \qquad (5)$$

where the first integral is evaluated over the body volume and the second over the surface. The quantity Q computed according to (5) is exactly the same as in the conventional theory. However, my calculations show that the volume contribution, $q^{(V)} \propto \mathrm{Im}(\mu\epsilon)$. I argue that in passive media, the second law of thermodynamics requires that $q^{(V)} > 0$ in contradiction with the inequality (1). In optically active media, it is possible to have $q^{(V)} < 0$ but the condition for negative refraction is then reversed and reads $\mathrm{Im}(\mu\epsilon) < 0$. Thus I come to the conclusion that negative refraction is not possible in either passive or active media.

The paper is organized as follows. In Section 2, I compute the volume contribution to the heating rate, $q^{(V)}$, for a monochromatic plane wave by two different methods and obtain two different expressions. In Section 3, I argue that the reason for this discrepancy is incorrect definition of the Poynting vector $\mathbf{S}$. When the correct definition (3) is adopted, the two methods yield the same result. Also, in Section 3, $q^{(V)}$ is computed for general monochromatic fields (not necessarily plane waves). In Section 4 I compute the surface contribution to the heating rate, $q^{(S)}$. Also in this section, the zero-frequency limit is discussed. In Section 5 I give a detailed proof that the second law of thermodynamics requires that $q^{(V)} > 0$. In Section 6, I discuss compatibility of the obtained expressions for the heating rate with the causality principle. In Section 7, I show that negative refraction is not possible even in anisotropic and nonlocal media. Finally, Sections 8 and 9 contain a discussion and a summary of obtained results.

2. Computation of the Heating Rate

The heating rate q is defined as the energy absorbed and transformed into heat by a material per unit volume (or surface, if there is a surface contribution), per unit time. In the case of oscillating electromagnetic fields, this energy must be averaged over time periods which are much larger than the characteristic period of oscillations. In this section, I use two different methods to compute q for a monochromatic plane wave propagating in a homogeneous, isotropic medium characterized by scalar and local (but time-dispersive) functions $\epsilon(\omega) = \epsilon'(\omega) + i\epsilon''(\omega)$ and $\mu(\omega) = \mu'(\omega) + i\mu''(\omega)$.

A. *First derivation of q*

First, we use the well-known formula for q which can be found in many standard textbooks, namely,

$$q^{(\mathrm{conv})} = \frac{1}{4\pi}\left\langle \mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t} + \mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t}\right\rangle \ , \qquad (6)$$

where $\langle \ldots \rangle$ denotes time averaging and the quantities $\mathbf{E}, \mathbf{D}$ and $\mathbf{H}, \mathbf{B}$ are the electric field and displacement and the magnetic field and induction, respectively. The superscript "(conv)" has been used to indicate that (6) gives the conventional result for the heating rate. In the case of a monochromatic field of frequency ω, (6) can also be written as

$$q^{(\mathrm{conv})} = \frac{\omega}{4\pi}\left[\epsilon''(\omega)\langle \mathbf{E}^2\rangle + \mu''(\omega)\langle \mathbf{H}^2\rangle\right] \ . \qquad (7)$$

Note that (6),(7) are quadratic in electromagnetic fields; correspondingly, $\mathbf{E}, \mathbf{D}, \mathbf{H}$ and $\mathbf{B}$ are defined in these expressions as real-valued quantities.

Let us take one step further and evaluate (7) for a plane wave propagating in a homogeneous medium. We shall seek an expression for the heating rate which contains only the amplitude of the electric, but not of the magnetic, field. To this end, we write

$$\mathbf{E} = \mathrm{Re}[\mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}] \ , \quad \mathbf{H} = \mathrm{Re}[\mathbf{H}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}] \ , \qquad (8)$$

where $\mathbf{E}_0$ and $\mathbf{H}_0$ are complex field amplitudes. Time averaging yields $\langle \mathbf{E}^2\rangle = (1/2)|\mathbf{E}_0|^2\exp(-2\mathbf{k}''\cdot\mathbf{r})$ and analogously for the magnetic field. Here $\mathbf{k}'' = \mathrm{Im}(\mathbf{k})$ and $\mathbf{k}$ satisfies $\mathbf{k}\cdot\mathbf{k} = \mu\epsilon(\omega/c)^2$. We now substitute the expressions for the time averages $\langle \mathbf{E}^2\rangle$ and $\langle \mathbf{H}^2\rangle$ in terms of the field amplitudes $\mathbf{E}_0$ and $\mathbf{H}_0$ into (7) to obtain

$$q^{(\mathrm{conv})} = \frac{\omega}{8\pi}\left[\epsilon''(\omega)|\mathbf{E}_0|^2 + \mu''(\omega)|\mathbf{H}_0|^2\right]e^{-2\mathbf{k}''\cdot\mathbf{r}} \ . \qquad (9)$$

Further, we want to express $|\mathbf{H}_0|^2$ in terms of $|\mathbf{E}_0|^2$. From the Maxwell equation $c\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t$ and from $\mathbf{B}_0 = \mu\mathbf{H}_0$, it follows that $\mathbf{H}_0 = [c/\omega\mu(\omega)]\mathbf{k}\times\mathbf{E}_0$. Therefore, $|\mathbf{H}_0|^2 = (c/\omega|\mu|)^2(\mathbf{k}\times\mathbf{E}_0)\cdot(\mathbf{k}^*\times\mathbf{E}_0^*) = (c/\omega|\mu|)^2[(\mathbf{k}\cdot\mathbf{k}^*)(\mathbf{E}_0\cdot\mathbf{E}_0^*) - (\mathbf{k}\cdot\mathbf{E}_0^*)(\mathbf{k}^*\cdot\mathbf{E}_0)]$. The wave vector of a propagating (that is, not evanescent) wave can always be written as $\mathbf{k} = k\hat{\mathbf{u}}$, where $\hat{\mathbf{u}}$ is a purely real unit vector such that $\hat{\mathbf{u}}\cdot\hat{\mathbf{u}} = 1$ and $k^2 = \mu\epsilon(\omega/c)^2$ is a complex scalar. In this case, $\mathbf{k}\cdot\mathbf{k}^* = |\mu(\omega)\epsilon(\omega)|(\omega/c)^2$ and $\mathbf{k}\cdot\mathbf{E}_0^* = \mathbf{k}^*\cdot\mathbf{E}_0 = 0$. The final expression for the heating rate then becomes

$$q^{(\mathrm{conv})} = \frac{\omega}{8\pi}\left[\epsilon''(\omega) + \frac{|\epsilon(\omega)|}{|\mu(\omega)|}\mu''(\omega)\right]|\mathbf{E}_0|^2 e^{-2\mathbf{k}''\cdot\mathbf{r}} \ . \qquad (10)$$

Already at this point we can notice that the coefficient in the parentheses in the above formula appears to be somewhat strange. Indeed, if μ is purely imaginary (e.g., near a resonance), this coefficient becomes $|\epsilon| + \epsilon''$. If,

in addition, $|\epsilon'| \gg \epsilon''$, the heating rate becomes proportional to $|\epsilon'|$.

We note that a propagating wave in an absorbing infinite medium grows exponentially in the direction $-\mathbf{k}''$. To avoid the unbounded growth, one has to consider a half space $z > 0$ into which an incident wave enters, e.g., from vacuum, and apply the condition $\hat{\mathbf{z}} \cdot \mathbf{k}'' > 0$. However, a wave which is refracted from vacuum into an absorbing medium is necessarily evanescent. This follows immediately from the fact that the projection of the wave vector on the plane $z = 0$ must be continuous at the interface and, therefore, purely real (since it is real in vacuum). In the case of evanescent waves, the equalities $\mathbf{k} \cdot \mathbf{k}^* = |\mu\epsilon|(\omega/c)^2$ and $\mathbf{k} \cdot \mathbf{E}_0^* = 0$ do not hold and the expression for $q^{(\mathrm{conv})}$ becomes more complicated. This effect is not important for weakly absorbing media and it will not be discussed here. We only note that the expression (18) which will be obtained below from the definition $q = \langle \mathbf{J} \cdot \mathbf{E} \rangle$ applies to both running and evanescent waves and, in any case, differs from (9) or (10).

B. *Second derivation of q*

We now compute the same quantity as in the previous subsection but using a different, presumably equivalent, definition. Namely, we write

$$q = \langle \mathbf{J} \cdot \mathbf{E} \rangle \ , \tag{11}$$

where $\mathbf{J}$ is the total current in the medium induced by the propagating electromagnetic field. We again emphasize that this current is formed by the charged particles (bound and conduction electrons, ions, etc.) which make up the medium. Equation (11) is simply the mathematical formulation of the statement that, in a stationary state, the heating rate is equal to the (time-averaged) work that the electric field exerts on the medium per unit time per unit volume.

We again consider a plane monochromatic wave with the electric field given by the first equation in (8). The current also has the form of a plane wave:

$$\mathbf{J} = \mathrm{Re}[\mathbf{J}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}] \ . \tag{12}$$

Note that the above formula is valid only inside the medium volume. At the boundary, there is an additional surface current related to magnetization. This current and the corresponding contribution to the heating rate will be considered separately in Section 4. We now focus on the volume contribution to the heating rate and denote the corresponding quantity by $q^{(V)}$. Time-averaging results in

$$q^{(V)} = \frac{1}{2}\mathrm{Re}(\mathbf{J}_0 \cdot \mathbf{E}_0^*) e^{-2\mathbf{k}''\cdot\mathbf{r}} \ . \tag{13}$$

To find $\mathbf{J}_0$, we write the two curl Maxwell equations as

$$c\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t \ , \quad c\nabla \times \mathbf{B} = \partial \mathbf{E}/\partial t + 4\pi \mathbf{J} \ , \tag{14}$$

where $\mathbf{J}$ is given by (2). Note that the above equations are equivalent to the usual macroscopic Maxwell equations if we define the auxiliary fields $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$ and $\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}$ and use (2). By taking the time derivative of the second equation in (14) and substituting $\partial\mathbf{B}/\partial t$ from the first equation, we find that

$$-4\pi\partial\mathbf{J}/\partial t = c^2\nabla \times \nabla \times \mathbf{E} + \partial^2\mathbf{E}/\partial t^2 \ . \tag{15}$$

At the next step, we substitute (12) and the first equation in (8) into (15) to obtain

$$\mathbf{J}_0 = -(c^2/4\pi i\omega)[\mathbf{k} \times \mathbf{k} \times \mathbf{E}_0 + (\omega/c)^2\mathbf{E}_0] \ . \tag{16}$$

We then use $\mathbf{k}\times\mathbf{k}\times\mathbf{E}_0 = -(\mathbf{k}\cdot\mathbf{k})\mathbf{E}_0$ and $\mathbf{k}\cdot\mathbf{k} = \mu\epsilon(\omega/c)^2$ (this holds for both propagating and evanescent waves) to further simplify the above expression for $\mathbf{J}_0$, which becomes

$$\mathbf{J}_0 = \frac{\omega}{4\pi i}[\mu(\omega)\epsilon(\omega) - 1]\mathbf{E}_0 \ . \tag{17}$$

Upon substitution of the above expression into (13), we arrive at

$$q^{(V)} = \frac{\omega|\mathbf{E}_0|^2}{8\pi}\mathrm{Im}\,[\mu(\omega)\epsilon(\omega)]\, e^{-2\mathbf{k}''\cdot\mathbf{r}} \ . \tag{18}$$

I shall generalize this result to the case of monochromatic field $\mathbf{E} = \mathrm{Re}[\mathbf{E}_\omega(\mathbf{r})\exp(-i\omega t)]$ (not necessarily a plane wave) in Eq. (30) below.

The expression (18) must be compared to (10). The respective formulae obviously differ. The reason for this discrepancy and the correct choice of the expression for q are discussed in Section 3.

C. *The two expressions for the heating rate and the constraints on ϵ and μ that follow from them*

If we accept the conventional result for the heating rate as correct, the second law of thermodynamics requires that, in a passive medium, $q^{(\mathrm{conv})} > 0$, where $q^{(\mathrm{conv})}$ is given for plane waves by (9) or (10). If, however, we assume that the alternative expression (18) is correct, then the second law requires that the volume contribution to the heating rate $q^{(V)}$ be positive (proof is given in Section 5). From this, a different constraint on the possible values of ϵ and μ is obtained.

We note right away that for the conventional expression (9) to be positive, it is not *necessary* that

$$\epsilon'' > 0 \ , \quad \mu'' > 0 \ , \tag{19}$$

although the above inequality is a *sufficient* condition. The *sufficient and necessary condition* is

$$|\mu|\epsilon'' + |\epsilon|\mu'' > 0 \ . \tag{20}$$

Neither (19) nor (20) prohibit negative refraction. We note that it can be argued (assuming (9) is correct) that

both inequalities $\epsilon'' > 0$ and $\mu'' > 0$ must hold simultaneously and independently to guarantee positivity of the heating rate [7,8]. Indeed, the condition (20) was derived for a plane wave and is, therefore, not the most general.

The alternative expression (18) imposes a different constraint on ϵ and μ. For $q^{(V)}$ given by the expression (18) to be positive, the *sufficient and necessary* condition is

$$\epsilon'\mu'' + \mu'\epsilon'' > 0 \ . \tag{21}$$

Thus, ϵ' and μ' can not be simultaneously negative while ϵ'' and μ'' are positive. In particular, (21) prohibits negative refraction in the sense that if the wave number k satisfies $k^2 = \epsilon\mu(\omega/c)^2$, its real and imaginary parts have the same sign, independently of the choice of the square root branch.

Finally, note the following interesting fact. If we put $\mu = 1$, formulae (10) and (18) become identical. But if we put $\epsilon = 1$ (e.g., in a purely magnetic material), the two expressions still differ. This is suggestive of the fact that magnetic losses are not properly accounted for in one of these formulae.

3. Correct Expressions for the Poynting Vector and the Heating Rate

Both formulae (10) and (18) do not obviously contradict any of the basic physical principles, such as the conservation laws. Therefore, we must choose the correct expression for q on less fundamental grounds. To this end, we examine the origin of the two definitions (6) and (11).

The expression (6) is obtained from $q = -\langle \nabla\cdot\mathbf{S}\rangle$ where the Poynting vector $\mathbf{S}$ is given by Eq. (4). Differentiation leads to

$$q = -\frac{c}{4\pi}\langle \mathbf{H}\cdot(\nabla\times\mathbf{E}) - \mathbf{E}\cdot(\nabla\times\mathbf{H})\rangle \ . \tag{22}$$

One then uses the macroscopic Maxwell equations to express $\nabla\times\mathbf{E}$ and $\nabla\times\mathbf{H}$ in terms of the corresponding time derivatives to arrive at (6). In the stationary case, when there is no accumulation of electromagnetic energy anywhere inside the medium, the formula $q = -\langle\nabla\cdot\mathbf{S}\rangle$ is undoubtedly correct. It is a mathematical expression of the statement that the total electromagnetic energy which enters into a small volume δV through its surface is entirely consumed to compensate for the irreversible (absorptive) losses in that volume.

The alternative formula, (11), is also a first-principles definition of the absorbed power per unit volume and appears to be unassailable. I emphasize again that the quantity $\mathbf{J}$ in (11) is the *total internal current* produced by all charged particles that compose the material. This includes bound electrons, conductivity electrons (if such are present), ions in the case of plasmas, etc.

So far, it appears that in either of the two approaches, the only formula that can be doubted is the definition of the Poynting vector (4) which was used to derive (6). It should be noted that in standard textbook expositions, the form the Poynting vector is postulated rather than derived. Thus, for example, Schwinger *et al.* (in *Classical Electrodynamics* [9, \$7.1]) consider the identity

$$\frac{c}{4\pi}\nabla\cdot(\mathbf{E}\times\mathbf{H}) + \frac{1}{4\pi}\left(\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} + \mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}\right) = 0 \tag{23}$$

which is trivially obtainable from the macroscopic Maxwell equations in the absence of external currents. Then Schwinger *et al.* write: “Our aim is to write this result as a local energy conservation law. We immediately identify, from the divergence term, the energy flux or Poynting vector $\mathbf{S}$ to be $\mathbf{S} = (c/4\pi)\mathbf{E}\times\mathbf{H}$.” The argument is, however, mathematically flawed. Indeed, one can take any scalar function $f(\mathbf{r}) \neq 0$ whose integral over the body volume is zero and write it as a divergence of a vector field, $f(\mathbf{r}) = \nabla\cdot\mathbf{F}(\mathbf{r})$, where $\mathbf{F}(\mathbf{r})$ vanishes outside of the body. One then can add $(c/4\pi)\nabla\cdot\mathbf{F}$ to the first term in the left-hand side of (23) and subtract it from the second term, and the identity will still hold. According to the logic of Refs. 9, one then has to define the Poynting vector as $\mathbf{S} = (c/4\pi)[\mathbf{E}\times\mathbf{H} + \mathbf{F}]$. Note that the field $\mathbf{F}$ does not need to be solenoidal, so that not only the definition of $\mathbf{S}$ is changed, but also of its divergence. This ambiguity in the conventional definition of $\mathbf{S}$ leads to a substantial strain. To quote Schwinger again, “... More intractable is the identification of the last term in (23).” I argue that such identification is, indeed, intractable because the term in question has no physical meaning.

A somewhat different approach to deriving the conventional expression for $\mathbf{S}$ is adopted by Landau and Lifshitz in *Electrodynamics of Continuous Medium* [10, \$80]. First, it is shown that Eq. (4) is valid in nonmagnetic media where $\mathbf{H} = \mathbf{B}$. Then Landau and Lifshitz argue that the normal component of $\mathbf{S}$ should be continuous when a wave crosses an interface between two media. Since the tangential components of both $\mathbf{E}$ and $\mathbf{H}$ are continuous, the normal component of $\mathbf{S}$ defined by (4) is continuous as well. Therefore, (4) should be valid in any media, including those with dispersion and a magnetic response.

I do not dispute here that the tangential components of $\mathbf{E}$ and $\mathbf{H}$ are continuous as long as there is no surface current at the interface which is formed by charges which are *external to the medium*. Note that sometimes such currents are referred to as “free currents” (even though they do not include the current of free electrons in the case of conductors). However, I claim that continuity of the normal component of $\mathbf{S}$ is an incorrect boundary condition for interfaces that separate two media at least one of which is magnetic. Indeed, it is known [10, \$29] that nonzero magnetization creates surface currents which are restricted to a very thin layer near the medium boundary. These currents are formed by the charges of the medium and, therefore, do not cause the tangential component of $\mathbf{H}$ to become discontinuous as follows immediately from the equation $c\nabla\times\mathbf{H} = \partial\mathbf{D}/\partial t$. When a wave crosses an

interface in which such surface current is flowing, a finite fraction of its energy is lost to the (positive or negative) work exerted by the electric field on the surface current. In this case, the normal component of $\mathbf{S}$ experiences a discontinuity. The role of the surface currents and their input to the heating rate is discussed in Section 4 below.

To obtain the correct expression for the Poynting vector, we start with the microscopic electric and magnetic fields, $\mathbf{e}$ and $\mathbf{h}$. The spatial averages of these fields are [10, \$1 and \$29] $\overline{\mathbf{e}} = \mathbf{E}$ and $\overline{\mathbf{h}} = \mathbf{B}$. Here the bar denotes spatial averaging over physically small volumes. Further, we write $\mathbf{e} = \mathbf{E} + \delta\mathbf{e}$ and $\mathbf{h} = \mathbf{B} + \delta\mathbf{h}$, where $\delta\mathbf{e}$ and $\delta\mathbf{h}$ are the fluctuating parts of the fields. The microscopic expression for the Poynting vector is

$$\mathbf{s} = \frac{c}{4\pi}\mathbf{e} \times \mathbf{h} \ . \tag{24}$$

We now average the above expression as follows:

$$\mathbf{S} \equiv \overline{\mathbf{s}} = \frac{c}{4\pi}\left[\ \mathbf{E} \times \mathbf{B} + \overline{\delta\mathbf{e}} \times \mathbf{B} + \mathbf{E} \times \overline{\delta\mathbf{h}} + \overline{\delta\mathbf{e} \times \delta\mathbf{h}}\ \right] \ . \tag{25}$$

By definition, $\overline{\delta\mathbf{e}} = \overline{\delta\mathbf{h}} = 0$. The term $\overline{\delta\mathbf{e} \times \delta\mathbf{h}}$ is quadratic in field fluctuations and can be omitted as small. It should be also noted that in materials which are random but isotropic on average, this term is identically zero by symmetry. We thus arrive at the expression (3) for the Poynting vector. We note that in vacuum, the Poynting vector is expressed in terms of the electric and magnetic fields. There is no conceivable physical reason why this should change if the field propagates through a material medium. But the average value of the magnetic field in the medium is $\mathbf{B}$, not $\mathbf{H}$. Despite the fact that $\mathbf{H}$ is commonly called the "magnetic field", it is actually an auxiliary quantity.

Let us adopt the definition (3) for the Poynting vector and compute $q^{(V)}$ from $q^{(V)} = -\langle \nabla \cdot \mathbf{S} \rangle$ for a general monochromatic fields of the form

$$\mathbf{E} = \mathrm{Re}\left[\mathbf{E}_\omega(\mathbf{r})e^{-i\omega t}\right] \ , \ \mathbf{D} = \mathrm{Re}\left[\mathbf{D}_\omega(\mathbf{r})e^{-i\omega t}\right] \ , \tag{26}$$
$$\mathbf{H} = \mathrm{Re}\left[\mathbf{H}_\omega(\mathbf{r})e^{-i\omega t}\right] \ , \ \mathbf{B} = \mathrm{Re}\left[\mathbf{B}_\omega(\mathbf{r})e^{-i\omega t}\right] \ , \tag{27}$$

where $\mathbf{D}_\omega = \epsilon(\omega)\mathbf{E}_\omega$, $\mathbf{B}_\omega = \mu(\omega)\mathbf{H}_\omega$, $c\nabla \times \mathbf{E}_\omega = i\omega\mathbf{B}_\omega$ and $c\nabla \times \mathbf{H}_\omega = -i\omega\mathbf{D}_\omega$. At the moment, we do not consider the heating rate at the surface where $\mathbf{S}$ has a discontinuity. We then have:

$$\langle \mathbf{S} \rangle = \frac{c}{8\pi}\mathbf{E}^*_\omega \times \mathbf{B}_\omega \tag{28}$$

and

$$\begin{aligned} q^{(V)} &= -\nabla \cdot \langle \mathbf{S} \rangle \\ &= \frac{c}{8\pi}\mathrm{Re}\left[-\nabla \cdot (\mathbf{E}^*_\omega \times \mathbf{B}_\omega)\right] \\ &= \frac{c}{8\pi}\mathrm{Re}\left[\mathbf{E}^*_\omega \cdot (\nabla \times \mathbf{B}_\omega) - \mathbf{B}_\omega \cdot (\nabla \times \mathbf{E}^*_\omega)\right] \\ &= \frac{c}{8\pi}\mathrm{Re}\left[\mathbf{E}^*_\omega \cdot (\nabla \times \mu(\omega)\mathbf{H}_\omega) + \frac{i\omega}{c}\mathbf{B}_\omega \cdot \mathbf{B}^*_\omega\right] \\ &= \frac{c}{8\pi}\mathrm{Re}\left[\frac{-i\omega}{c}\mu(\omega)\mathbf{E}^*_\omega \cdot \mathbf{D}_\omega\right] \\ &= \frac{\omega|\mathbf{E}_\omega|^2}{8\pi}\mathrm{Im}[\mu(\omega)\epsilon(\omega)] \ . \end{aligned} \tag{29}$$

We thus have derived the following formula for $q^{(V)}$:

$$q^{(V)} = \frac{\omega|\mathbf{E}_\omega|^2}{8\pi}\mathrm{Im}[\mu(\omega)\epsilon(\omega)] \ . \tag{30}$$

If we set $\mathbf{E}_\omega = \mathbf{E}_0\exp(i\mathbf{k}\cdot\mathbf{r})$, the above expression coincides with formula (18) which was derived previously from the definition $q^{(V)} = \langle \mathbf{J} \cdot \mathbf{E} \rangle$ for the case of a plane wave of the form (8).

Next, consider the definition $q^{(V)} = \langle \mathbf{J} \cdot \mathbf{E} \rangle$. We have already used this definition to compute $q^{(V)}$ for a plane wave with the result given by Eq. (18). Now we repeat the calculation for more general monochromatic fields (26),(27). The current (except at the medium surface) can also be written in a similar form, namely,

$$\mathbf{J} = \mathrm{Re}\left[\mathbf{J}_\omega(\mathbf{r})e^{-i\omega t}\right] \ , \tag{31}$$

where

$$\mathbf{J}_\omega = \frac{1}{4\pi}\left(c\nabla \times \mathbf{B}_\omega + i\omega\mathbf{E}_\omega\right) \ . \tag{32}$$

We then write

$$\begin{aligned} q^{(V)} &= \langle \mathbf{J} \cdot \mathbf{E} \rangle = \frac{1}{2}\mathrm{Re}(\mathbf{J}_\omega \cdot \mathbf{E}^*_\omega) \\ &= \frac{1}{8\pi}\mathrm{Re}\left[i\omega\mathbf{E}_\omega \cdot \mathbf{E}^*_\omega + c(\nabla \times \mathbf{B}_\omega) \cdot \mathbf{E}^*_\omega\right] \\ &= \frac{c}{8\pi}\mathrm{Re}\left[\mu(\omega)(\nabla \times \mathbf{H}_\omega) \cdot \mathbf{E}^*_\omega\right] \\ &= \frac{c}{8\pi}\mathrm{Re}\left[\frac{-i\omega}{c}\mu(\omega)\epsilon(\omega)\mathbf{E}_\omega \cdot \mathbf{E}^*_\omega\right] \\ &= \frac{\omega|\mathbf{E}_\omega|^2}{8\pi}\mathrm{Im}[\mu(\omega)\epsilon(\omega)] \ . \end{aligned} \tag{33}$$

The result of this calculation coincides with (30).

Thus, we can conclude that if $\mathbf{S}$ is defined by (3), the two definition of the heating rate, $q^{(V)} = -\langle \nabla \cdot \mathbf{S} \rangle$ and $q^{(V)} = \langle \mathbf{J}\cdot\mathbf{E} \rangle$ are equivalent and we have the statement of local energy conservation which, in the stationary case, reads

$$\langle \mathbf{J} \cdot \mathbf{E} \rangle + \langle \nabla \cdot \mathbf{S} \rangle = 0 \ . \tag{34}$$

4. Heating Rate at the Surface and the Total Absorbed Heat

The derivation of the heating rate form the formula (11) was so far restricted to points inside the medium. In this section, the additional surface term $q^{(S)}$ is derived.

First, consider the definition (11). In the monochromatic case, the current in this formula is given by (32) where $\mathbf{B}_\omega = \mu(\omega)\mathbf{H}_\omega$. In the sequence of equalities (33), I have, at one point, replaced the term $\nabla \times \mathbf{B}_\omega$ by $\mu(\omega)\nabla \times \mathbf{H}_\omega$. This operation is only valid inside the medium volume. Close to the surface, we must write

$$\nabla \times \mathbf{B}_\omega = \nabla \times \mu(\mathbf{r})\mathbf{H}_\omega = \mu(\mathbf{r})\nabla \times \mathbf{H}_\omega + [\nabla\mu(\mathbf{r})] \times \mathbf{H}_\omega \ , \tag{35}$$

where the dependence of μ on position has been indicated explicitly. In the case of macroscopically homogeneous media, $\nabla\mu(\mathbf{r}) = 0$ everywhere except at the surface, where $\mu(\mathbf{r})$ experiences a discontinuity. If we restrict attention to points $\mathbf{r}$ which are on the surface, evaluation of $\nabla\mu(\mathbf{r})$ results in the additional surface current [10, \$29]

$$\mathbf{J}_\omega^{(S)} = -c\hat{\mathbf{n}} \times \mathbf{M}_\omega \ , \tag{36}$$

where $\hat{\mathbf{n}}$ is the outward unit normal to the boundary at the point $\mathbf{r}$. Note that the definition of the surface current (36) does not contain a spatial delta-function and that $\mathbf{J}_\omega^{(S)}$ has different physical units then the volume current $\mathbf{J}$.

We now find the surface contribution to the heating rate as

$$q^{(S)} = \frac{1}{2}\mathrm{Re}\left(\mathbf{J}_\omega^{(S)} \cdot \mathbf{E}_\omega^*\right) \ . \tag{37}$$

A straightforward derivation results in

$$q^{(S)} = \frac{c}{8\pi}\mathrm{Re}\left[(1-\mu)(\hat{\mathbf{n}} \times \mathbf{H}_\omega) \cdot \mathbf{E}_\omega^*\right] \ . \tag{38}$$

It is also possible to start from the definition $q = -\nabla \cdot \mathbf{S}$, take into account the fact that the normal component of $\mathbf{S}$ defined by (3) experiences a discontinuity at the medium boundary, and arrive, in a straightforward manner at

$$q^{(S)} = \frac{c}{8\pi}\mathrm{Re}\left[(1-\mu)(\mathbf{H}_\omega \times \mathbf{E}_\omega^*) \cdot \hat{\mathbf{n}}\right] \ . \tag{39}$$

Since $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$, the two expressions (38) and (39) are identical.

The total heat absorbed by the body, Q, is given by Eq. (5). It is easy to see that this quantity is the same as in the conventional theory. Indeed, Q can be computed by integrating the energy flux through any surface enclosing the body. Such surface can be drawn in free space where the conventional expression for the Poynting vector (4) and the expression derived in this paper (3) coincide. Therefore, the proposed change in the form of the Poynting vector and of the heating rate does not affect any of the previously established results for differential or integral cross sections, such as the Mie formulae for extinction, absorption and scattering cross sections of spheres.

The surface contribution to the heating rate derived in this section requires several additional comments. The obvious distinction between the surface term $q^{(S)}$ (39) and the volume term $q^{(V)}$ (30) is that the volume term is proportional to the frequency ω while the surface term is not. Of course, it is incorrect to say that $q^{(V)}$ always vanishes in the zero-frequency limit because $\lim_{\omega\to 0}[\omega\epsilon(\omega)\mu(\omega)] = 4\pi i\sigma\mu(\omega = 0)$, where σ is the static conductivity of the material [11]. However, it appears that the surface term does not vanish in the zero-frequency limit even if we formally set $\sigma = 0$. This possibility is worrisome and is discussed below.

First, this paper is primarily concerned with the high-frequency superficial magnetism which originates due to the loop-like conductivity currents flowing in elementary cells of composite materials. The magnetic susceptibility of such composites identically vanishes in the zero-frequency limit [12, 13]. Moreover, the formula (11) in which the current is given by (2) assumes that all currents obey the classical laws of motion. This may not be the case when magnetization is caused by spin aligning, as in the cases of ferro- and para-magnetism. However, even in the case of ferromagnetism, the current $c\nabla\times\mathbf{M}$ is a macroscopic quantity. According to the Ehrenfest theorem, all macroscopic quantities obey the classical laws of motion. Nevertheless, it should be acknowledged that some of the phenomena associated with ferromagnetism, such as the hysteresis, are clearly outside of the theoretical frame of the classical electrodynamics of continuous media. Such effects can be accounted for phenomenologically but not in a fully self-consistent way.

Thus, the zero-frequency limit might not be the proper test for the theory developed in this paper. I will, however, argue that it is possible to apply this theory to the zero-frequency limit without obtaining unphysical effects or contradictions. To this end, I consider below two simple examples.

The first example is a straight ferromagnetic or paramagnetic cylindrical wire of radius a, conductivity σ and permeability μ carrying a current of uniform density J directed along the axis of the wire. I disregard here the Hall effect that results in a non-uniform current distribution over the wire cross section [14]. The quantities μ and σ are purely real at zero frequency. Then my theory predicts that the volume will be heated at the rate (per length L of the wire)

$$Q^{(V)}/L = \pi a^2\sigma\mu E^2 \tag{40}$$

and the surface will be heated or cooled at the rate

$$Q^{(S)}/L = \pi a^2\sigma(1-\mu)E^2 \ . \tag{41}$$

In a ferro- and paramagnetic materials, $\mu > 1$ (for the case of ferromagnetic, the nonlinearity of the magnetization curve and the magnetic memory of the material must be taken into account, which is not a trivial task),

so that the surface term is negative. But the total heat produced in the system is the sum of both contributions, namely,

$$Q/L = Q^{(V)}/L + Q^{(S)}/L = \pi a^2 \sigma E^2 \ , \qquad (42)$$

which is the Joule's law. In a steady state, the overall flux of thermal energy through the wire surface, which is the experimentally measurable quantity (e.g., in a calorimeter) is given by Q/L, in agreement with the Joule's law.

The second example is a magnetic object placed in crossed external electric and magnetic fields. For simplicity, consider a long cylinder uniformly magnetized along its axis. The magnetization will create loop-like surface currents that flow around the cylinder axis. We now place the cylinder in an external electric field which is perpendicular to the cylinder axis (this example has been previously considered by Pershan [15]). If the cylinder is conducting (as are most ferromagnets), the tangential component of the electric field at the cylinder surface, as well as the electric field inside the cylinder, vanish and we obtain $q^{(S)} = q^{(V)} = 0$, as expected. A somewhat more complicated situation arises if we formally set $\sigma = 0$ and $\mu - 1 \neq 0$. The volume term $q^{(V)}$ is still zero in this case, but the surface term $q^{(S)}(\varphi)$ may become locally nonzero (here ρ, z, φ are the cylindrical coordinates). Even though it can be easily seen that

$$\int_0^{2\pi} q^{(S)}(\varphi) d\varphi = 0 \ , \qquad (43)$$

we still expect no local heating or cooling of the surface in the static equilibrium. The contradiction is resolved by noting that the state of the cylinder described above can not be its true state of equilibrium and that the initial assumption $\sigma = 0$ and $\mu - 1 \neq 0$ was unphysical. This assumption contradicts the mechanical equilibrium of charges that make up the circular magnetization currents. Classically, these charges rotate with constant angular velocity around the cylinder axis due to a phenomenological radial force. It is, however, not possible to introduce a phenomenological restoring *tangential* force such as the harmonic restoring force in the Lorentz model of dielectrics. Indeed, such restoring tangential force will preclude the magnetization current from flowing in the first place. Consequently, imposition of an external tangential force (due to the external electric field) in the absence of a tangential restoring force is bound to break the equilibrium of the system. Specifically, the external electric field will cause electric charge to accumulate on the cylinder surface until the tangential component of the electric field is completely nullified. The resultant state will be the true static equilibrium of the system. The conclusion is that magnetized objects can not have identically zero conductivity. Of course, the value of σ can be small, but so is usually the value of $\mu - 1$. Another important consideration is that, in addition to conductivity, magnetics also have some dielectric response whose effect is to diminish the tangential electric field at the body surface.

The example considered above suggests that the conventional definition of the Poynting vector (4) is erroneous because it predicts existence of an equilibrium state which contradicts mechanical stability of the system.

5. Thermodynamic Considerations and Impossibility of Negative Refraction

Many authors believe that the unique properties of the negative refraction materials originate from the fact that the phase velocity and the Poynting vector in such media are oppositely directed. This property is sometimes referred to as "backward propagation". For example, to quote Marques *at al.* [16, $1.2], "...most of the surprising unique electromagnetic properties of these media arise from this backward propagation property." If the expression (3) for the Poynting vector is correct, as I argue in this paper, then the phase velocity and the Poynting vector always point in the same direction and "backward propagation" is impossible, at least in electromagnetically homogeneous media. I will, however, apply more fundamental thermodynamic considerations to show that the inequality (1) is physically prohibited, regardless of whether it results in those "surprising unique effects" or not.

To this end, it is instructive to introduce the "accessible heat". This is the heat (either positive or negative) which can be transferred from the body to a heat reservoir on a time scale which is short compared to the time scales associated with heat diffusion in the body. Obviously, this is the heat generated at the surface. Let

$$Q^{(S)} \equiv \oint_S q^{(S)}(\mathbf{r}) d^2 r = Q_+^{(S)} - Q_-^{(S)} \ . \qquad (44)$$

Here $Q_+^{(S)}$ is obtained by integration over the surface areas where $q^{(S)}(\mathbf{r})$ is positive and $Q_-^{(S)}$ is obtained by integration over the surface areas where $q^{(S)}(\mathbf{r})$ is negative. Let us further assume that the material exhibits negative refraction and $q^{(V)}(\mathbf{r})$ is negative, so the heat generated in the volume,

$$Q^{(V)} \equiv \int_V q^{(V)}(\mathbf{r}) d^3 r \qquad (45)$$

is also negative. Then we have

$$Q = -|Q^{(V)}| + Q_+^{(S)} - Q_-^{(S)} \qquad (46)$$

or, equivalently,

$$Q_+^{(S)} = Q + |Q^{(V)}| + Q_-^{(S)} > Q \ . \qquad (47)$$

Thus, the positive accessible heat is greater than the total heat absorbed in the body. I will now demonstrate that this contradicts the Carnot theorem and, moreover, can be used to create a *perpetuum mobile* of the second kind. To see that this is, indeed the case, consider the cyclic process shown in the Fig. 1.

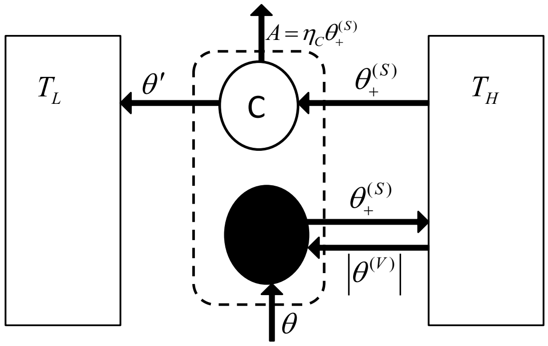

Fig. 1. A cyclic process involving a negative refractive index material that violates the Carnot theorem. The black oval represents a negative refraction medium and the white oval is an ideal Carnot engine.

In this cycle, the following events happen: (1) A negative-refraction sample represented by the black oval (referred to as the "body" below) at the initial temperature T_H is irradiated for a period of time Δt which is short compared to the time scales associated with heat diffusion in the body, yet long compared to the electromagnetic oscillations period, so that radiation is almost monochromatic. The body absorbs the energy $\theta = Q\Delta t$ from the radiation field. (2) The body is brought in contact with a heat reservoir at the temperature T_H which has very high heat conductivity; the amount of thermal energy $\theta_+^{(S)} = Q_+^{(S)}\Delta t$ generated at the body's surface is transferred adiabatically to this reservoir. (3) The body is disconnected from the reservoir and heat diffusion takes place in the body until the equilibrium temperature $T' < T_H$ is reached. (4) An ideal Carnot engine is operated for one cycle between the hot reservoir and a colder reservoir whose temperature is $T_L < T_H$. The Carnot engine absorbs the heat $\theta_+^{(S)}$ from the hot reservoir, makes useful work $A = \eta_C\theta_+^{(S)}$ and rejects some amount of heat $\theta' = \theta_+^{(S)} - A$ to the cold reservoir. Here

$$\eta_C = 1 - T_L/T_H \tag{48}$$

is the efficiency of an ideal Carnot engine. (5) The body is again brought in contact with the hot reservoir; now the heat $|\theta^{(V)}|$ flows back from the hot reservoir to the body. In the end of this process, the body has the temperature T_H. (6) We disconnect the body from the hot reservoir. Now the cycle is complete and the system has returned to its original state. Note that steps (1) and (2), as well as (3) and (4) can be combined, i.e., occur simultaneously.

The net effect of the above thermodynamic transformation is the following: The electromagnetic field has done the work θ on the body which was immediately dissipated into heat θ; we then converted this heat into the useful work A. The overall efficiency of this process is

$$\eta = \frac{A}{\theta} = \eta_C \frac{\theta_+^{(S)}}{\theta_+^{(S)} - |\theta^{(V)}|} = \eta_C \frac{1}{1 - |\theta^{(V)}|/\theta_+^{(S)}} > \eta_C \tag{49}$$

in violation of the Carnot theorem. Moreover, we can operate a *perpetuum mobile* of the second kind if $A > \theta$ or, equivalently, $\eta > 1$. This is achieved if $T_L/T_H < |\theta^{(V)}|/\theta_+^{(S)}$. There is no physical reason why this condition can not be met. In particular, it can be met quite easily in the case of low-loss negative refraction materials such that $|\theta^{(V)}|/\theta_+^{(S)} = 1 - \delta$ where $\delta \ll 1$. Then even relatively small temperature difference $T_H - T_L$ would be sufficient to extract more energy from the heat reservoir than was absorbed from the electromagnetic field. Note that in order to obtain the contradiction, it is essential that the "accessible" heat $\theta_+^{(S)}$ be larger than the total absorbed heat θ. This is always the case for negative refraction materials, as was shown above. Therefore, I conclude that negative refraction is impossible.

It is also possible to use negative refraction to construct a refrigeration cycle in violation of the Carnot theorem.

6. Heating Rate and Causality

As is well known, the causality principle requires that $\epsilon(\omega)$ and $\mu(\omega)$ have no singularities in the upper half-plane when viewed as functions of the complex variable ω. In the lower complex half-plane, these functions may have singularities. We will assume now that all such singularities are simple poles. If, in addition, we account for the symmetry property $\epsilon(-\omega) = \epsilon^*(\omega)$ and $\mu(-\omega) = \mu^*(\omega)$, we can write ϵ and μ in the most general form as

$$\epsilon(\omega) = 1 + 4\pi\chi_e(\omega) \ , \quad \mu(\omega) = 1 + 4\pi\chi_m(\omega) \ , \tag{50}$$

where

$$\chi_e(\omega) = \sum_k \frac{f_k(\omega)}{a_k^2 - \omega^2 - i\alpha_k\omega} \ , \tag{51}$$

$$\chi_m(\omega) = \sum_k \frac{g_k(\omega)}{b_k^2 - \omega^2 - i\beta_k\omega} \ . \tag{52}$$

Here χ_e and χ_m are the electric and magnetic susceptibilities, respectively, a_k, b_k, α_k and β_k are coefficients and $f_k(\omega)$ and $g_k(\omega)$ are analytical functions of the frequency which have no singularities and satisfy $f_k(-\omega) = f_k^*(\omega)$ and analogously for g_k. The representation (50)-(52) is customary in the theory of dispersion. In the electric

case, the functions f_k are usually positive constants interpreted as oscillator strengths. In the magnetic case, the typical form of $g_k(\omega)$ is $g_k(\omega) = c_k\omega^2$ where the coefficients c_k can be negative, i.e., in the case of diamagnetics. Terms proportional to ω^{2n} with $n > 1$ in the Taylor expansion of $f_k(\omega)$ and $g_k(\omega)$ are not usually considered because of the physical requirement that both ϵ and μ are bounded when $\omega \to \infty$. The physical interpretation of the remaining constants appearing in (51),(52) is as follows: a_k and b_k are the resonance frequencies and α_k and β_k are the respective relaxation constants.

We wish to examine whether the representation (50)-(52) is compatible with the inequality $q(\omega) \propto \mathrm{Im}[\epsilon(\omega)\mu(\omega)] > 0$ which must hold for all positive frequencies. Obviously, the latter imposes some constraints on the coefficients appearing in formulae (51)-(52). However, it is easy to show that coefficients that satisfy $q(\omega) > 0$ for all $\omega > 0$ do exist. For instance, a sufficient condition for $q > 0$ is obtained when one of the susceptibilities is significantly smaller than the other. Let us write

$$\mathrm{Im}(\epsilon\mu) = 4\pi \mathrm{Im}(\chi_e + \chi_m) + (4\pi)^2 \mathrm{Im}(\chi_e\chi_m) \ . \tag{53}$$

Since $(4\pi)^2\mathrm{Im}(\chi_e\chi_m) > -(4\pi)^2|\chi_e\chi_m|$ and $\mathrm{Im}(\chi_e + \chi_m) > \mathrm{Im}(\chi_e)$, a sufficient condition for $\mathrm{Im}(\mu\epsilon) > 0$ is $|\chi_m| < \chi_e''/4\pi|\chi_e|$. This inequality can always be satisfied in the whole frequency range for sufficiently small coefficient c_k (assuming $g_k(\omega) = c_k\omega^2$). Analogously, $q > 0$ if $|\chi_e| < \chi_m''/4\pi|\chi_m|$.

Thus, we have obtained two sufficient conditions for $q > 0$. However, these conditions are not necessary. To derive a condition which is both sufficient and necessary, consider the special case of a single electric and single magnetic resonance with positive and frequency-independent oscillator strengths, f_e^2 and g_e^2. That is, assume that ϵ and μ are given by

$$\epsilon(\omega) = 1 + 4\pi\frac{f_e^2}{\omega_e^2 - \omega^2 - i\gamma_e\omega} \ , \tag{54}$$

$$\mu(\omega) = 1 + 4\pi\frac{f_m^2}{\omega_m^2 - \omega^2 - i\gamma_m\omega} \ , \tag{55}$$

where all coefficients are positive. Then a straightforward calculation shows that

$$\frac{A_e(\omega)A_m(\omega)}{4\pi\omega}\mathrm{Im}(\mu\epsilon) = a\omega^4 + b\omega^2 + c \ , \tag{56}$$

where

$$\begin{aligned} a = &\quad f_e^2\gamma_e + f_m^2\gamma_m \ , & (57)\\ b = &\quad f_m^2\gamma_m(\gamma_e^2 - 2\omega_e^2) + f_e^2\gamma_e(\gamma_m^2 - 2\omega_m^2) \\ &\qquad - 4\pi f_e^2 f_m^2(\gamma_e + \gamma_m), & (58)\\ c = &\quad 4\pi f_e^2 f_m^2(\gamma_m\omega_e^2 + \gamma_e\omega_m^2) \\ &\qquad + f_m^2\gamma_m\omega_e^4 + f_e^2\gamma_e\omega_m^4 & (59)\end{aligned}$$

and $A_e(\omega), A_m(\omega)$ are positive factors defined by

$$A_e(\omega) = (\omega_e^2 - \omega^2)^2 + (\gamma_e\omega)^2 \ , \tag{60}$$

$$A_m(\omega) = (\omega_m^2 - \omega^2)^2 + (\gamma_m\omega)^2 \ . \tag{61}$$

The right-hand side of (56) is a quadratic polynomial in ω^2 with a positive free term c. If we assume that relaxation constants are small so that $\gamma_e^2 < 2\omega_e^2$ and $\gamma_m^2 < 2\omega_m^2$, the coefficient b is negative. In this case, the necessary and sufficient condition that the polynomial is positive is $\mathcal{D} = b^2 - 4ac < 0$. A tedious but straightforward calculation yields the following expression for the discriminant $\mathcal{D}$:

$$\begin{aligned} \frac{\mathcal{D}}{\gamma_e\gamma_m} = &\quad \gamma_e\gamma_m\left[f_m^4(\gamma_e^2 - 4\omega_e^2) + f_e^4(\gamma_m^2 - 4\omega_m^2)\right] \\ &+ 2f_e^2 f_m^2\left[\gamma_e^2\gamma_m^2 - 2(\gamma_m^2\omega_e^2 + \gamma_e^2\omega_m^2)\right. \\ &\qquad \left. - 2(\omega_e^4 + \omega_m^4) + 4\omega_e^2\omega_m^2\right] \\ &- 8\pi f_e^2 f_m^2\left\{f_m^2\left[\gamma_e^2 + \gamma_e\gamma_m + 2(\omega_m^2 - \omega_e^2)\right]\right. \\ &\qquad \left. + f_e^2\left[\gamma_m^2 + \gamma_m\gamma_e + 2(\omega_e^2 - \omega_m^2)\right]\right\} \\ &+ (4\pi)^2 f_e^4 f_m^4(\gamma_e + \gamma_m)^2 \ . \end{aligned} \tag{62}$$

Assuming that $b < 0$, the sufficient and necessary condition for $\mathrm{Im}(\mu\epsilon) > 0$ is that the above expression is negative. This, of course, leads to a very complicated inequality. However, if the relaxation constants are small compared to all other physical scales of the problem, this inequality can be simplified and reads

$$\begin{aligned} &(\omega_e^2 - \omega_m^2)^2 + 4\pi(\omega_m^2 - \omega_e^2)(f_m^2 - f_e^2) \\ &\qquad > 4\pi^2 f_e^2 f_m^2 \frac{(\gamma_e + \gamma_m)^2}{\gamma_e\gamma_m} \ . \end{aligned} \tag{63}$$

To illustrate how inequality (63) works, I have plotted $\mathrm{Im}(\mu\epsilon)$ as a function of ω for two different sets of parameters (see Fig. 2 caption for details). As can be seen, if the parameters satisfy (63), $\mathrm{Im}(\mu\epsilon) > 0$ in the whole frequency range (Fig. 2a). If, however, we use a set of parameters that does not satisfy (63), there appears a frequency range in which $\mathrm{Im}(\mu\epsilon) < 0$ (Fig. 2b). As I have argued above, $q \propto \mathrm{Im}(\mu\epsilon)$. Therefore, the set of parameters used to compute the curve shown in Fig. 2b leads to negative heating rate in the frequency range indicated by the horizontal arrow. These values of parameters violate the second law of thermodynamics and, therefore, can not be realized in any material, either natural or artificial.

Thus, we have seen that the expression for the heating rate derived in this paper does not contradict causality. However, it imposes constraints on the possible values of constants in the dispersion formulae (51),(52). Generally, these constraints are very complicated mathematically. In the simplest case of one magnetic and one electric resonance (ϵ and μ given by formulae (54) and (55)), the condition is that the discriminant (62) is negative.

In the limit of small relaxation constants γ_e and γ_m, this condition can be approximated by the much more simple inequality (63).

We finally note that the condition (63) is only sufficient but not necessary if the relaxation is so strong that the coefficient b given by formula (59) is positive. Also, in the case when one of the resonance frequencies is zero, as is the case for electric permittivity of conductors, inequality (63) may become an inaccurate approximation of the more general inequality $\mathcal{D} < 0$ where $\mathcal{D}$ is given by (62).

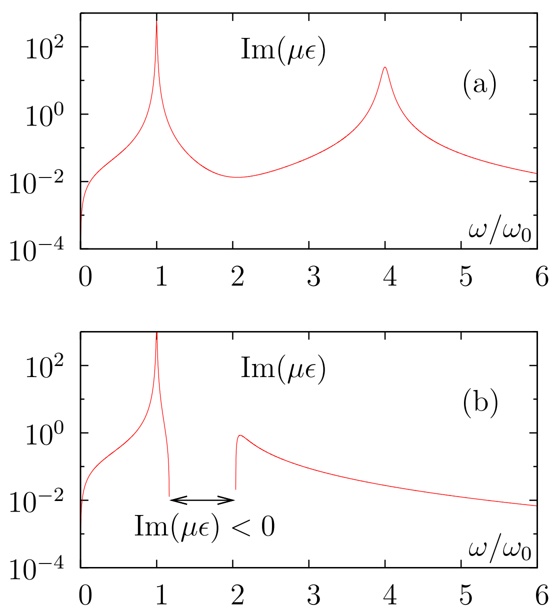

Fig. 2. Plots of $\mathrm{Im}[\mu(\omega)\epsilon(\omega)]$ where ϵ and μ are given by (54),(55) for different sets of parameters. Plot (a): $\omega_m = \omega_0$, $\omega_e = 4\omega_0$, $f_m = 0.5\omega_0$, $f_e = \omega_0$, $\gamma_m = 0.01\omega_0$ and $\gamma_e = 0.1\omega_0$; ω_0 is an arbitrary frequency scale. The parameters satisfy inequality (63). Plot (b): same parameters as in plot (a) but $\omega_e = 2\omega_0$. With this change, (63) is no longer satisfied in the frequency range denoted by the horizontal arrow. Negative values of $\mathrm{Im}[\mu(\omega)\epsilon(\omega)]$ are not shown in the plots due to the use of logarithmic scale.

7. Heating Rate in Anisotropic and Nonlocal Media

The general case of a medium with magnetic and electric anisotropy and nonlocality is quite complicated. The wave vector $\mathbf{k}$ of a plane wave propagating in such a medium can be found from the following condition:

$$\det\left|\hat{\epsilon}^{-1}\mathbf{k}\times\hat{\mu}^{-1}\mathbf{k}\times\ +\left(\frac{\omega}{c}\right)^2\right| = 0 \qquad (64)$$

where $\hat{\epsilon} = \hat{\epsilon}(\omega, \mathbf{k})$ and $\hat{\mu} = \hat{\mu}(\omega, \mathbf{k})$ are $\mathbf{k}$-dependent tensors.

The dispersion relation (64) is simplified for the case of propagating waves. A *propagating* (as opposed to an *evanescent*) wave is characterized by a wave vector $\mathbf{k} = k\hat{\mathbf{u}}$ where k is a complex scalar and $\hat{\mathbf{u}}$ is a purely real unit vector such that $\hat{\mathbf{u}}\cdot\hat{\mathbf{u}} = 1$. Thus, a propagating wave can, in principle, experience spatial decay. The important point is that, in the propagating case, the wave vector is completely characterized by a direction in space (the unit vector $\hat{\mathbf{u}}$) and by a single scalar k. We can utilize this property to rewrite (64) as

$$\det\left|-k^2\hat{T} + \left(\frac{\omega}{c}\right)^2\right| = 0\ , \qquad (65)$$

where $\hat{T} = -\hat{\epsilon}^{-1}\hat{\mathbf{u}}\times\hat{\mu}^{-1}\hat{\mathbf{u}}\times$. In general, the 3×3 tensor $\hat{\mathbf{T}}$ is symmetric but not Hermitian. Therefore, its eigenvectors and eigenvalues, denoted here by $\mathbf{v}_j$ and $1/\tau_j$, are complex. For each direction $\hat{\mathbf{u}}$, the wave number of a propagating wave is determined from one of the equations $k^2 = \tau_j(\omega/c)^2$ while the polarization of the j-th mode is given by $\mathbf{E}_0 = a\mathbf{v}_j$, a being an arbitrary complex constant.

Interestingly, it is possible to make a statement about the restrictions that are imposed by the condition $q^{(V)} > 0$ on the wave number k without explicitly solving the dispersion equations. We note that the formula $q = \langle\mathbf{J}\cdot\mathbf{E}\rangle$ is valid in any electromagnetically homogeneous media. We then consider monochromatic, propagating plane wave with the wave vector $\mathbf{k}$, so that the fields are of the form

$$\mathbf{E} = \mathrm{Re}\left[\mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right]\ , \qquad (66)$$

$$\mathbf{B} = \mathrm{Re}\left[\mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right]\ , \qquad (67)$$

$$\mathbf{J} = \mathrm{Re}\left[\mathbf{J}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right]\ , \qquad (68)$$

and

$$4\pi\mathbf{J}_0 = i\omega\mathbf{E}_0 + ic\mathbf{k}\times\mathbf{B}_0\ . \qquad (69)$$

We then obtain

$$q^{(V)} = \frac{ce^{-2\mathbf{k}''\cdot\mathbf{r}}}{8\pi}\mathrm{Im}\left[(\mathbf{k}\times\mathbf{B}_0)\cdot\mathbf{E}_0^*\right]\ . \qquad (70)$$

We now use $\mathbf{k}\times\mathbf{B}_0 = (c/\omega)\mathbf{k}\times\mathbf{k}\times\mathbf{E}_0$ and the identity $\mathbf{a}\times\mathbf{b}\times\mathbf{c} = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})$ to arrive at the following result:

$$q^{(V)} = \frac{\omega e^{-2\mathbf{k}''\cdot\mathbf{r}}}{8\pi(\omega/c)^2}\mathrm{Im}\left[|\mathbf{E}_0|^2(\mathbf{k}\cdot\mathbf{k}) - (\mathbf{k}\cdot\mathbf{E}_0)(\mathbf{k}\cdot\mathbf{E}_0^*)\right]\ . \qquad (71)$$

Note that we have not used any constitutive relations in the derivation of (71). Also note that the wave vector $\mathbf{k}$ must satisfy the dispersion relation (64). If $\mathbf{k} = k\hat{\mathbf{u}}$, it is always possible to write $(\mathbf{k}\cdot\mathbf{E}_0)(\mathbf{k}\cdot\mathbf{E}_0^*) = \cos^2\theta|\mathbf{E}_0|^2$, where θ is a purely real angle. From this, we obtain the final expression for q:

$$q^{(V)} = \frac{\omega|\mathbf{E}_0|^2 e^{-2\mathbf{k}''\cdot\mathbf{r}}}{8\pi} \frac{\sin^2\theta \ \mathrm{Im}(k^2)}{(\omega/c)^2} \ . \qquad (72)$$

The phenomenon of negative refraction requires that the direction in which a wave exponentially decays due to absorption in the medium is opposite to its phase velocity. Mathematically, this means that the real and imaginary parts of the complex wave number must have opposite signs. But for this to be true, it is required that $\mathrm{Im}(k^2) < 0$. However, Eq. (72) implies that $\mathrm{Im}(k^2) > 0$. Thus, negative refraction is not physically attainable even in anisotropic and nonlocal media.

Finally, we note that one can formally choose the polarization and the wave number in such a way that $\sin^2\theta = 0$ so that the medium does not absorb radiation. However, it is easy to see that waves with $\sin^2\theta = 0$ do not satisfy the dispersion relation (64).

8. Discussion

In this section, I address certain anticipated objections to the theory developed in this paper, as well as discuss some of its limitations.

The first and the most obvious objection is that there have been a number of works which claim experimental demonstration of negative refraction in electromagnetically homogeneous materials. Such experiments can be classified into two kinds. The direct-kind experiments measure the deflection of a beam passing through an experimental sample made of a subwavelength-structured "metamaterial". However, in most experiments of this kind, the linear size of the smallest metamaterial element, ℓ is not much smaller than even the vacuum wavelength λ. Strictly speaking, ℓ should be compared to the wavelength *inside* the material. Further, the more physically relevant parameter is $k\ell = 2\pi\ell/\lambda$. In typical experimental demonstrations of negative refraction, this parameter is of the order of unity. Under these circumstances, interpretation of the experimental results in terms of the bulk constants ϵ and μ is problematic. There are also experiments in which the negative refraction is measured indirectly by means of measuring the transmission and reflection coefficients t and r of a subwavelength-structured thin film. Here, as in the case of direct-kind experiments, it is very difficult to achieve $k\ell \ll 1$. Additionally, the indirect-kind experiments rely on phase measurements of high-frequency electromagnetic fields and on a analytical procedure of extracting ϵ and μ from the measurements of t and r. Both of these tasks are notoriously difficult and have recently been subject to some controversy [17–19].

The second objection is based on the factually incorrect, yet widespread belief that the magnetic field can, under certain circumstances, do work, i.e., on magnetic moments. Theory developed in this paper is based on the premise that only electric field can do work. The question of whether the magnetic force can do work is simultaneously simple and complicated. Of course, it immediately follows from the expression for the Lorentz force that the magnetic force does no work on a moving charged particle. Yet, apart from this simple observation which can be found in most textbooks on classical electrodynamics, there has been almost no serious discussion of this question in scientific literature. At the same time, situations in which the magnetic force is *apparently* doing work are quite abundant. Recently, the question was addressed in a mathematically rigorous way by Deissler [20]. In this reference, it is shown that the magnetic force does no work on a classical magnetic moment under any circumstances. It is further shown that the magnetic force does no work on an atom where magnetization is due to orbital angular momentum. Finally, Deissler shows that there is a fully self-consistent description of the quantum spin in which the magnetic field does no work either.

Third objection is based on the belief that composite media can be assigned *effective* medium parameters which describe (approximately) some phenomena associated with wave propagation through such media but not the others. I believe that such contention was expressed, for example, by Simovski [19], although implicitly. My reply to this is that most experimentally measurable quantities, such as the intensity, are bilinear in the electric and magnetic fields. Therefore, any useful homogenization model must correctly predict such quadratic combinations, including the Poynting vector and the heating rate.

Fourth objection is that the "magnetic" current $c\nabla\times\mathbf{M}$ is somehow different in its physical properties from the "electric" current $\partial\mathbf{P}/\partial t$ and, therefore, obeys different laws of motion. Of course, in the case of metamaterials, both currents have exactly the same physical origin. But even in the most general case, both currents are macroscopic and there is no valid physical basis to apply different laws of motion to them. It is also not possible to do so mathematically. Assume that we know a vector field $\mathbf{J}(\mathbf{r})$. Assume also that we know that $\mathbf{J}(\mathbf{r}) = \mathbf{J}_e(\mathbf{r}) + \mathbf{J}_m(\mathbf{r})$ where $\mathbf{J}_e(\mathbf{r}) = \partial\mathbf{P}(\mathbf{r})/\partial t$ and $\mathbf{J}_m(\mathbf{r}) = c\nabla\times\mathbf{M}(\mathbf{r})$. Is it possible to find uniquely $\mathbf{J}_e$ and $\mathbf{J}_m$ if we know $\mathbf{J}$ (but not $\mathbf{P}$ or $\mathbf{M}$)? It is known that $\mathbf{J}_m$ is solenoidal. If it were also known that $\mathbf{J}_e$ is irrotational, we would be able to use the Helmholtz theorem to uniquely decompose $\mathbf{J}$ into the irrotational and the solenoidal parts corresponding to $\mathbf{J}_e$ and $\mathbf{J}_m$. But the only instance when $\mathbf{J}_e$ is irrotational is the static case. Therefore, beyond strict statics, there is no unique way to disentangle the term $c\nabla\times\mathbf{M}$ from the total current $\mathbf{J}$.

The fifth set of objections is related to the zero-frequency limit. This limit is discussed in detail in Section 4. Here I would like to reiterate the following. The theory developed in this paper is based on the fundamental assumption that all currents obey the same classical laws of motion. Although the author sees no physical reason for this assumption to be untrue even in the zero frequency limit, reasonable caution must be exercised when applying the results to magnetization caused by quan-

tum spins. It is theoretically possible that the magnetic susceptibility which is due to spin alignment has different physical and mathematical properties when compared to the magnetic susceptibility which is due to classical eddy currents. In this case, the total permeability must be written as $\mu - 1 = (\mu_{\text{classical}} - 1) + (\mu_{\text{quantum}} - 1)$, where μ_{quantum} is the contribution to the total permeability due to quantum effects which are manifest only at low frequencies, and μ should be replaced by $\mu_{\text{classical}}$ in the expressions for the heating rate derived in this paper.

Finally, the essential requirement for applicability of the results derived in this paper is that the medium is electromagnetically homogeneous or can be effectively homogenized. Mathematically, this means that the medium must support running plane waves as its electromagnetic modes. This condition is not satisfied in photonic crystals and similar structures which support propagating modes in the form of *Bloch waves*.

9. Conclusions

The article has the following conclusions: (i) The correct definition of the Poynting vector in magnetic media is (3). If this definition is used, then the local energy conservation law in the form (34) holds, where $\mathbf{J}$ is given by (2). (ii) The heating rate $q^{(V)}$, defined as the amount of energy converted to heat per unit time per unit volume, is proportional to the factor $\mathrm{Im}(\mu\epsilon)$ *inside the volume occupied by the material*; heating rate at the surface is given by Eqs. (38) or (39) in Section 4. (iii) It follows from the detailed thermodynamic considerations of Section 5 that negative refraction contradicts the second law of thermodynamics. This statement holds for active or passive media and in the presence of anisotropy and spatial dispersion, as is shown in Section 7.

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